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Brief communication

Effect of shear-layer thickness on the Strouhal–Reynolds number relationship for bluff-body wakes

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Abstract

In a previous paper, we provided a rationale for the empirically observed St–Re number relationship for vortex shedding in bluff-body wakes. This rationale derives from a mechanism of vortex formation observed in numerical simulations coupled with an estimate of the terms in the vorticity transport equation based on this mechanism. Adopting the typical size of the body D as the characteristic length scale resulted in a rationale which matches the traditional $1/\mathrm{Re}$-fit. Here, we propose to adopt the thickness of the separated shear layers as the length scale which governs the diffusion of vorticity during the vortex-formation process instead of D. Thus, providing a new rationale matching Williamson–Brown's $1/\sqrt{\mathrm{Re}}$-fit, which has one order of magnitude less error than the traditional fits in terms of $1/\mathrm{Re}$.

Keywords: Vortex streets; Separated flows; Vortex dynamics

1. Introduction

During the past five decades, there have been extensive measurements which yielded different coefficients for the Strouhal–Reynolds number relationship for the laminar regime of vortex shedding in bluff-body wakes. Those fits generally follow the lead of Roshko (1954), who plotted the parameter $\mathrm{Ro} = \mathrm{St}\,\mathrm{Re} = fD^2/\nu$ (where f is the shedding frequency and D the characteristic length of the body) versus Re. Ro is now known as the Roshko number. Roshko (1954) found a linear least-squares fit for the Ro–Re plot, which gives a St–Re relationship in terms of $1/\mathrm{Re}$,

$$\mathrm{St} = A - B/\mathrm{Re}. \tag{1}$$

Following Roshko's work, many curves of the St–Re relation were published, often showing little agreement between them, and a controversy started about the nature and place of the several jump discontinuities in the data that were observed. This long-running debate was largely resolved by Williamson (1988a) who found that manipulating the end boundary conditions to enforce parallel shedding, the resulting St–Re curve can be made continuous throughout the laminar range ($49 < \mathrm{Re} < 178$). It is now believed that this universal parallel-shedding curve represents measurements for purely two-dimensional vortex shedding (Williamson, 1989).

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More recently, Fey et al. (1998) showed visually that a plot of St versus $1/\sqrt{\mathrm{Re}}$ for the laminar regime resembles a straight line. The same year, Williamson and Brown (1998) put forward a new St–Re relationship for the cylinder wake in terms of $1/\sqrt{\mathrm{Re}}$ which has one order of magnitude less error than the traditional fits in terms of 1/Re. Following the same line of reasoning as Lord Rayleigh, who suggested that St can be expressed in terms of a Taylor's expansion of 1/Re (Rayleigh, 1915), Williamson and Brown (1998) proposed a series expansion in terms of $1/\sqrt{\mathrm{Re}}$

$$\mathrm{St} = \left(A + B/\sqrt{\mathrm{Re}} + C/\mathrm{Re} + \cdots\right). \tag{2}$$

By truncating the series (2) to the first two terms, Williamson and Brown (1998) found a least-squares fit for the laminar shedding regime using the parallel-shedding data from Williamson and Brown (1988a, 1989). They obtained the expression $\mathrm{St} = 0.2665 + 1.018/\sqrt{\mathrm{Re}}$ and compared it with the traditional two-term fit $\mathrm{St} = 0.2175 + 5.106/\mathrm{Re}$ for the same experimental data. Fig. 1, taken from Williamson and Brown (1998), shows that the "$\sqrt{\mathrm{Re}}$-formula" lies much closer to the experimental data than the traditional 1/Re-fit, giving an average fitting-error of 0.0006 compared with the 0.0021 of the traditional formula. As shown by Wang et al. (2000) present data collapse quite well on the Williamson and Brown (1998) curve, except for the data near the critical stage. The deviations between the Williamson and Brown's two-term equation and the Wang et al. (2000) data for the unheated cylinder are less than 1% in the whole laminar Re range.

Williamson and Brown (1998) also proposed a connection between the $1/\sqrt{\mathrm{Re}}$-fit and two length scales: the wake width (L^*), and the vorticity-thickness of the separated shear layers (δ_ω). They give a physical interpretation of the $\sqrt{\mathrm{Re}}$-formula in which the constant term A is due to the size or physical shape of the body itself, while the following terms in powers of $1/\sqrt{\mathrm{Re}}$ are associated with the shear layer thickness.

Theoretical models for the St–Re number relationship have also been introduced. Ahlborn et al. (2002) proposed a phenomenological model for vortex-street formation downstream of a bluff body based on the analysis of the mass, momentum and energy balance, giving relationships between the Strouhal, the drag coefficients and the Reynolds number. Roushan and Wu (2005) proposed a new St–Re relation based on the observations of the structure of a vortex street in flowing soap films, suggesting a two-parameter form $\mathrm{St} = 1/(A + B/\mathrm{Re})$ to describe laminar vortex shedding. In a previous paper (Ponta and Aref, 2004), we put forward that the empirical St–Re fit is quite natural and follows readily from an elucidation of the vortex formation mechanism observed in numerical simulations and an estimate of the terms in the vorticity transport equation. This estimate was based on the adoption of D as the characteristic length scale which governed the physical processes involved. That resulted in a rationale which matches the traditional 1/Re-fit. Here, we propose to adopt δ_ω instead of D as the length scale which governs the diffusion of vorticity across the separated shear layers; thus, providing a rationale based on the vortex-formation mechanism which endorses the $1/\sqrt{\mathrm{Re}}$-fit.

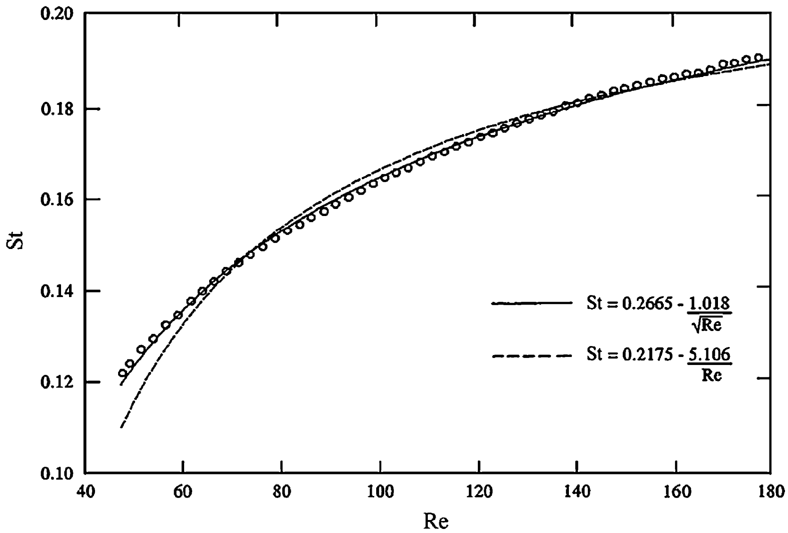

Fig. 1. Comparison of the two-term "$\sqrt{\mathrm{Re}}$-formula" versus the traditional 1/Re-fit for the experimental data from Williamson (1988a, 1989). [Taken from Williamson and Brown (1998)].

2. A $1/\sqrt{\mathrm{Re}}$-rationale for the St–Re relationship

In our model, we assumed that the shedding period is somehow related to the time needed to "nucleate" a vortex, which will subsequently take its place in the vortex street wake. Eddies are not shed directly from the cylinder but are formed downstream by the wake instability. This is accomplished by a gradual roll-up of free shear layers emanating from the cylinder (Kovasznay, 1949). It is clear that the roll-up plays an essential role in vortex formation. The key idea of our work is the decomposition of the velocity field obtained from DNS computations in its solenoidal and harmonic components. This decomposition allows us to clearly identify the eddy structures the shear layers are rolling-up around during the process of formation of the vortex cores.

Identifying the eddy structures represents a challenging issue. The topology of the velocity field strongly depends on the choice of frame of reference. If the observer moves with the cylinder, incipient eddy structures can be observed in the vicinity of the solid surface, while the far wake shows wavy streamlines and no eddies. Conversely, if the observer is fixed in the laboratory, the typical pattern of streamlines associated with a vortex street appears in the far wake, but the pattern of streamlines in the vicinity of the cylinder appear distorted [a good example can be seen in Batchelor (2000, plate 11)]. Thus, the choice of frame of reference influences the observations and any conclusions regarding mechanism. Moreover, for an observer moving with the cylinder, it appears that the eddies are advected downstream from the low-speed zone in the vicinity of the cylinder until they reach a steady advection regime in the far wake. Thus, to describe the streamline pattern properly, we need to find a frame of reference that follows each eddy as it accelerates in its travel downstream. To simply choose some moving frame *by hand* would be really complicated and the observations would be strongly biased, so we proposed another solution. We can always decompose the incompressible velocity field (in a frame of reference moving with the cylinder) as follows $\boldsymbol{u} = \boldsymbol{u}_v + \boldsymbol{v}$, where $\boldsymbol{u}_v$ is solenoidal and has the same curl as $\boldsymbol{u}$, and $\boldsymbol{v}$ is the irrotational and solenoidal (i.e., harmonic) component. We put forward that this task of advection, accelerating the eddies to their final state of motion in the vortex street wake itself, is accomplished by the irrotational, solenoidal part, $\boldsymbol{v}$, of the full velocity field $\boldsymbol{u}$. In other words, we propose to assign to $\boldsymbol{v}$ this task of advection of the eddy structures defined by the streamline pattern in $\boldsymbol{u}_v$. Thus, whilst $\boldsymbol{v}$ is responsible for the advection of the vortex structures as a whole, $\boldsymbol{u}_v$ is responsible for the advection of vorticity inside the vortex structure, which gives the *internal* rearrangement of vorticity within the vortex core. For prescribed velocity conditions on the boundary of the analysis domain, $\boldsymbol{v}$ is uniquely determined (Batchelor, 2000, Section 2.7). Now $\boldsymbol{v}$ (which is easy to compute) and then $\boldsymbol{u}_v = \boldsymbol{u} - \boldsymbol{v}$ are both uniquely determined.

After identifying the eddy structure the shear layers are rolling-up around, the physical mechanisms of vorticity transport that operate inside this eddy structure are analyzed. In Fig. 2 we see a schematized version of the process where the shear layer of positive vorticity starts to roll-up around the incipient eddy on the lower right of the body (in this case, a cylinder) until the core of this eddy has produced a (roughly) homogeneous distribution of vorticity

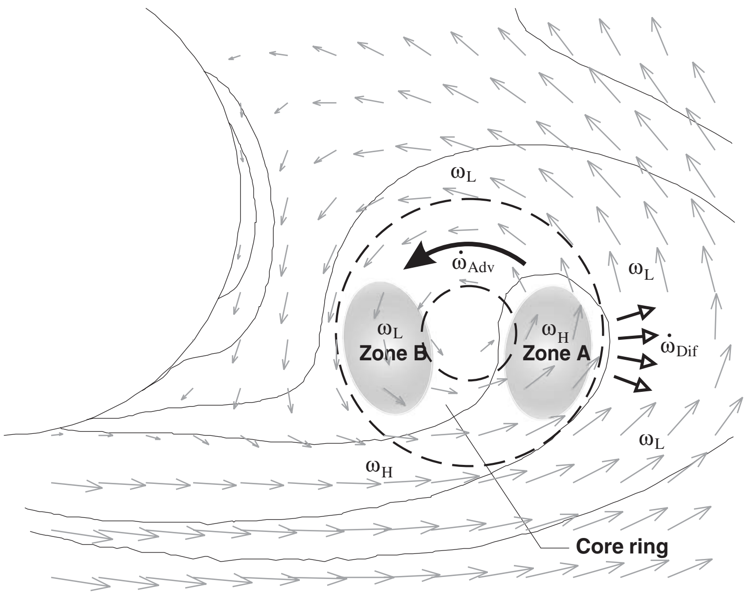

Fig. 2. Schematic view of the roll-up of a vortex-core.

all around its periphery. The amount of time required to form this homogeneous core of vorticity is, in essence, half a period of the shedding cycle. We shall use this schema to estimate the various terms appearing in the vorticity transport equation (for two-dimensional flow),

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \omega = \nu \nabla^2 \omega. \tag{3}$$

Referring to Fig. 2 the homogenization is carried on by the transport of vorticity along the periphery of the core from the high-vorticity zone at the head of the rolling shear layer (zone A) to the low-vorticity zone at the opposite side of the core ring (zone B). Two mechanisms act simultaneously: advection, which takes place mainly around the core, and diffusion, which acts mainly in the radial direction outward from the core. These two mechanisms have opposing effects on the homogenization. While advection is trying to *build up* the core, diffusion tends to *spread out* the vorticity before it can arrive at zone B.

In terms of (3), a suitable estimate for the rate of change of vorticity at zone B is $(\omega_H - \omega_L)f$, where f is the shedding frequency. We move the remaining advective derivative to the right-hand side and estimate it by $U(\omega_H - \omega_L)/\ell$, where U is the free stream speed, and ℓ is a characteristic length scale associated to the radius of the vortex-core ring for which we adopted the diameter of the cylinder D. This term gives the rate of intrinsic rearrangement (homogenization) within the core. We see that the advective velocity must generally point opposite to the gradient of vorticity so the term acts as a source term for vorticity build-up. Finally, the diffusive sink of vorticity ($\nu\nabla^2\omega = \nu\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\omega$) produces a term that may be estimated as $-\nu(\omega_H - \omega_L)/(\ell\delta_\omega)$, where δ_ω is the length scale related with the vorticity gradient taken across the shear layer and ℓ is the length scale related with the divergence. There is strong evidence that $\delta_\omega/D \sim 1/\sqrt{\mathrm{Re}}$. The separating shear-layer thickness δ will depend on the growth of the boundary layer on the forward part of the cylinder which, subject to a boundary layer approximation, gives $\delta/D \sim 1/\sqrt{\mathrm{Re}}$ (Williamson and Brown, 1998). Bloor (1964) also assumed this relationship and showed that the shear layer instability frequency scaled approximately with $1/\sqrt{\mathrm{Re}}$. Williamson and Brown (1998) include measured values of δ_ω for the laminar and other regimes together for Re up to 1200, showing close comparison with the formula $\delta_\omega/D = 4.217/\sqrt{\mathrm{Re}}$. Then, collecting these estimates in an equation by introducing two dimensionless constants, k_a for the advective process and k_d for the diffusive process, we obtain

$$(\omega_H - \omega_L)f = k_a U \frac{(\omega_H - \omega_L)}{D} - k_d \nu \frac{(\omega_H - \omega_L)}{D^2}\sqrt{\mathrm{Re}}. \tag{4}$$

Simple algebra then gives

$$\frac{fD}{U} = k_a - k_d \frac{\nu}{UD}\sqrt{\mathrm{Re}}, \tag{5}$$

or, in terms of the Strouhal and Reynolds numbers,

$$\mathrm{St} = k_a - \frac{k_d}{\sqrt{\mathrm{Re}}}. \tag{6}$$

3. Concluding remarks

It is significant that (6) matches not only the expression for the two-term $\sqrt{\mathrm{Re}}$-formula in the laminar regime, but also the $\sqrt{\mathrm{Re}}$-fit to the experimental data in the *mild* turbulent regime known as "Mode B" (Williamson and Brown, 1998; Williamson, 1988b). Mode B covers what is also known as the TrW2 (upper transition-in-wake) and TrSL1 (lower transition-in-shear-layers) regimes (Zdravkovich, 1997). Mode B is characterized by the appearance of fine-scale streamwise vortex structures in the near wake. The transition from laminar shedding to Mode B is located between $\mathrm{Re} = 230$ and 260. At $\mathrm{Re} = 260$, the primary wake instability behaves remarkably like the laminar shedding mode, with the exception of the presence of the fine-scale streamwise vortex structures (Williamson, 1996). As Re is then increased, the fine-scale three dimensionality becomes increasingly disordered.

In Mode B, the eddies are turbulent, and this change in the flow is likely to change the values of both scaling constants k_a and k_d from those obtained for the laminar regime, but the scaling itself appears to remain valid. For long-span cylinders we have many of these fine-scale streamwise vortex structures distributed along the span [the spanwise length scale is around one diameter (Williamson, 1996)]. This allows us to interpret their effect in a span-averaged way. Thus, even though Mode B is clearly a three-dimensional phenomenon, our two-dimensional model appears to cover it as well. Williamson and Brown (1998) applied the $\sqrt{\mathrm{Re}}$-formulation to the frequency data for Mode B finding that $\mathrm{St} = 0.2234 + 0.3490/\sqrt{\mathrm{Re}}$ gives an average error-of-fit of only 0.0005. Fig. 3 [taken from Williamson and Brown (1998)] shows how the St–Re data are essentially represented very well by two $\sqrt{\mathrm{Re}}$-formulae up to (at least) $\mathrm{Re} = 1200$.

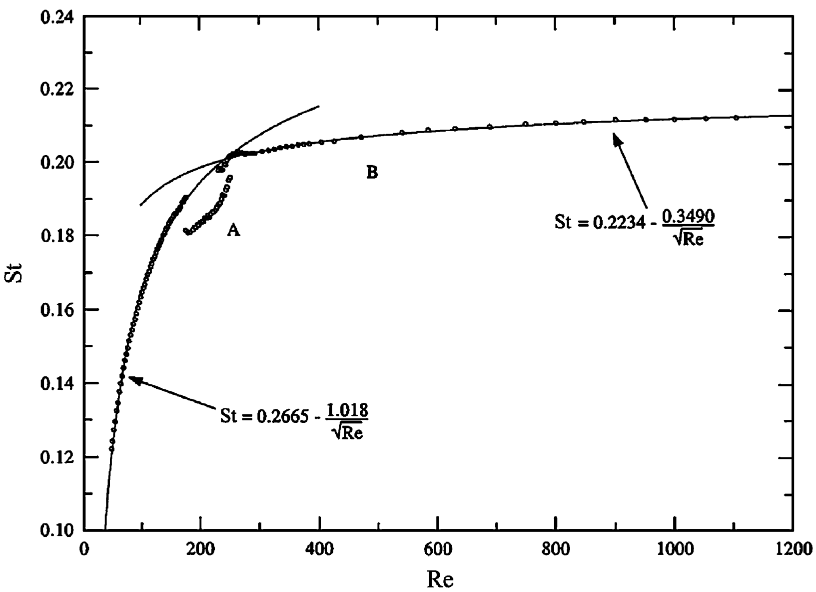

Fig. 3. Fit of $\sqrt{\mathrm{Re}}$-formulae to the laminar and mild turbulent regimes up to Re = 1200. [Taken from Williamson and Brown (1998)].

Between Re $\approx$ 178 and 260, the primary shedding vortices become unstable to a spanwise waviness [termed as Mode A (Williamson, 1988b)], corresponding with curve A in Fig. 3. The application of our model to Mode A, also known as TrW1 (lower transition-in-wake) regime (Zdravkovich, 1997), was treated in detail in Ponta and Aref (2004).

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